

Lecture 1

Text Book = Wiley International edition, 3rd edition, Elementary Linear Algebra
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Chapter 1

System of Linear Equations and Matrices

[Eq. of st. Line $ax + by + c = 0 \iff$ Linear equations

Ex. 1 Linear or not:

$$3x^2 + y + 7x = 0$$

(NL)

$$x + xy + y = 7$$

(NL)

$$x + y = \sin x$$

(NL)

$$x + 4y + c = 0$$

(L)

$$x + y = \sin \frac{\pi}{3}$$

(L)

$$x + \left(\frac{1}{x}\right) + y = 7$$

(NL)

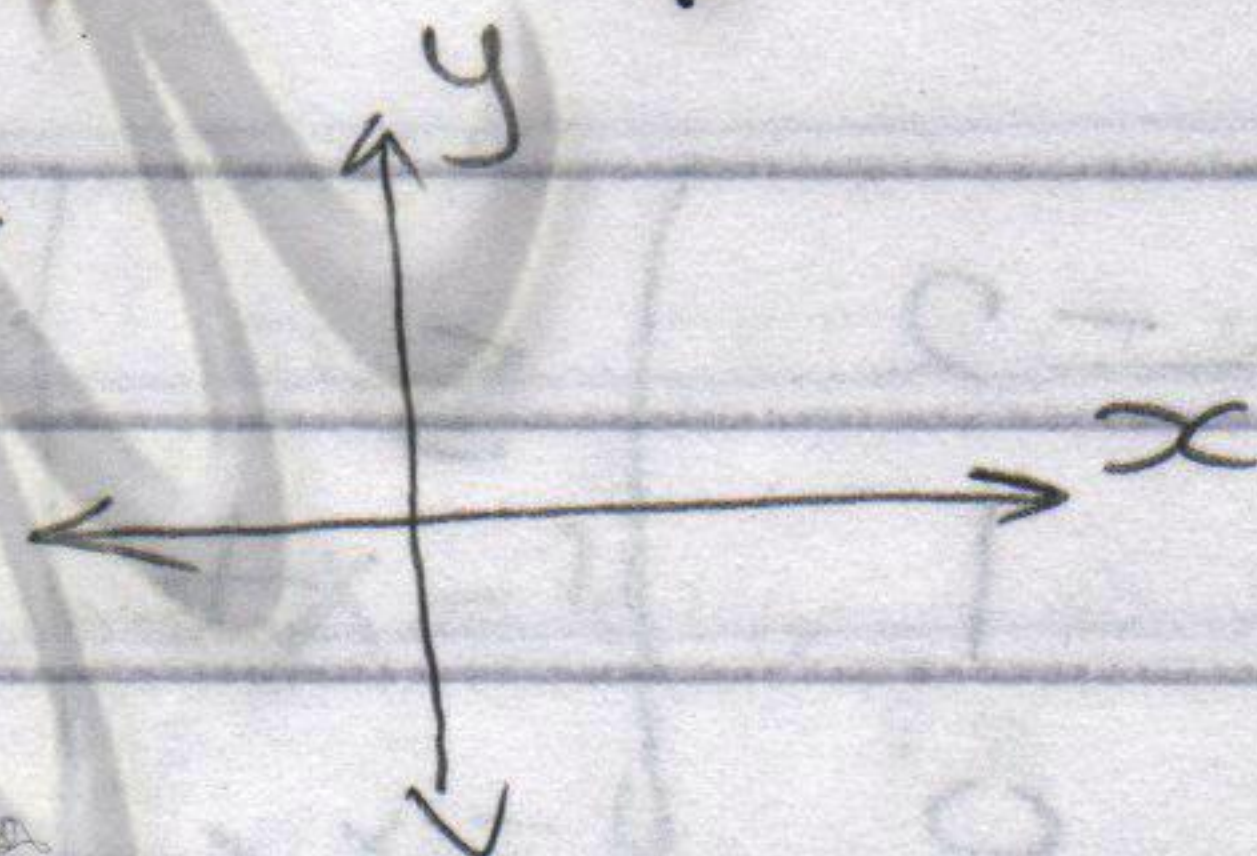
(1.1) Introduction to Systems of Linear Equations:

A System of Linear Equations

A st. Line in xy -Plane

$$a_1x + b_1y = b$$

is a Linear equation where a_1, b_1, b are real Const.
Arbitrary Const.

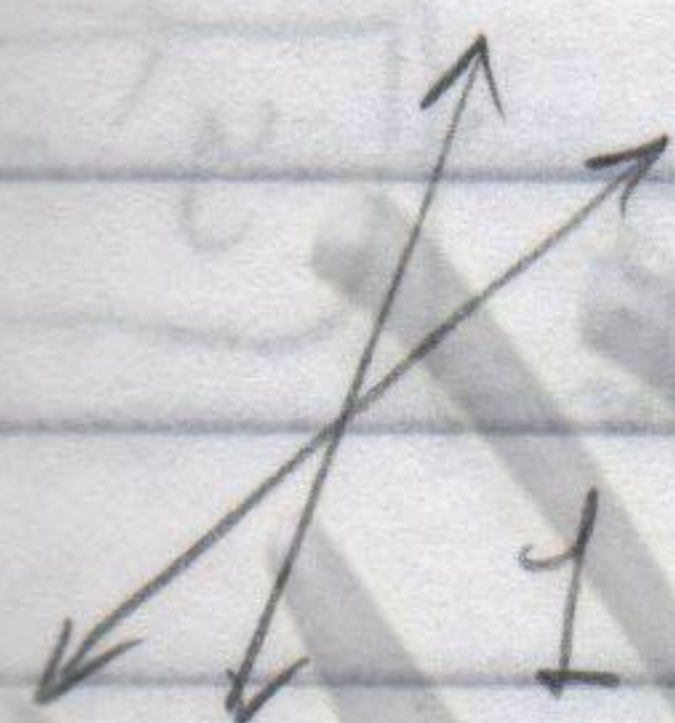


The Linear Equation in the n Variables

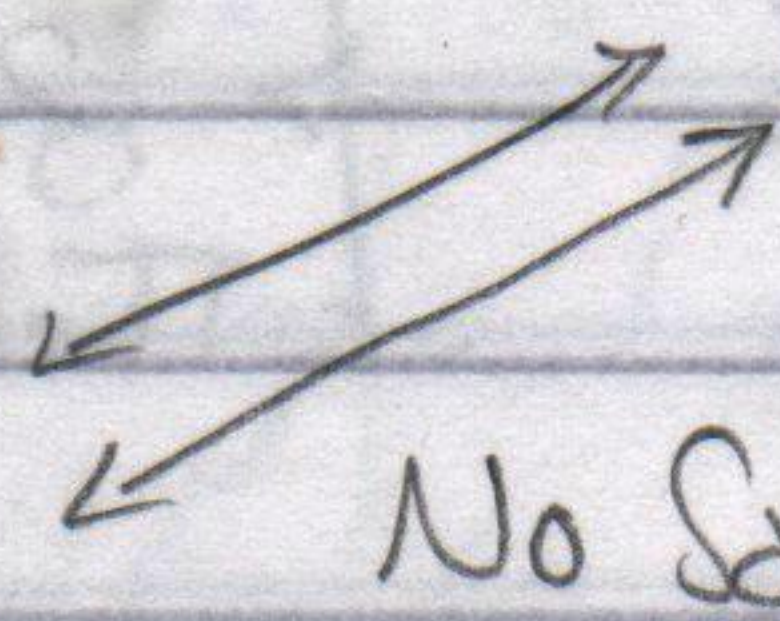
$$x_1, x_2, \dots, x_n$$

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

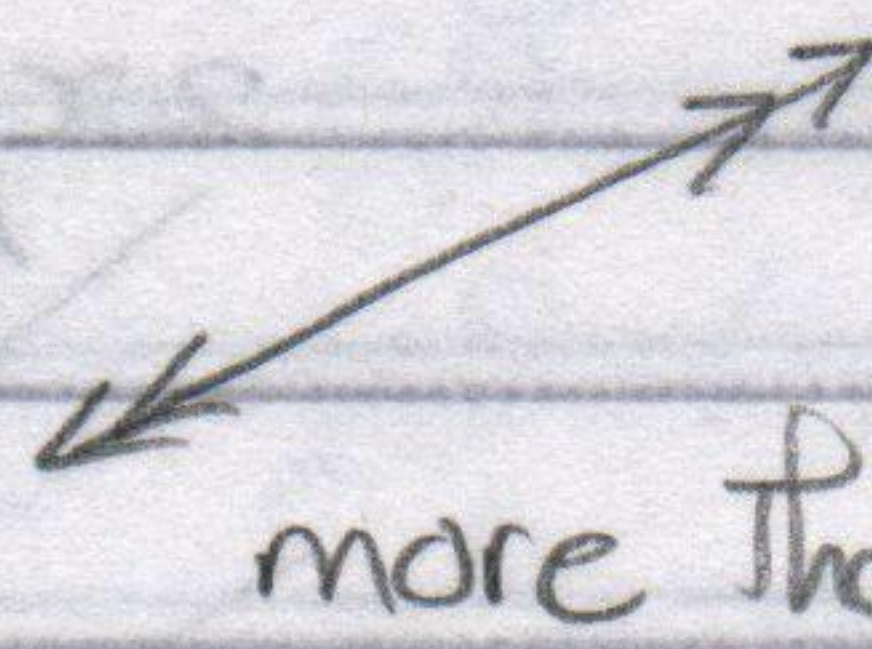
where $a_1, a_2, a_3, \dots, a_n, b$ are arbitrary Const.



1 Solu.



No Solu



more than one Solu.

Ex. 1 Linear or not

$$x + 3y = 7$$

(L)

$$y = \frac{1}{2}x + 3z + 1$$

(L)

$$x_1 - 2x_2 - 3x_3 + x_4 = 7$$

(L)

$$x + 3\sqrt{y} = 0$$

(NL)

$$3x + 2y - z + xz = 4$$

(NL)

$$y = \sin x$$

(NL)

$$3\sin x + 4\sin y + 7\sin z = 6$$

(L)

Soln. Of Linear Equations

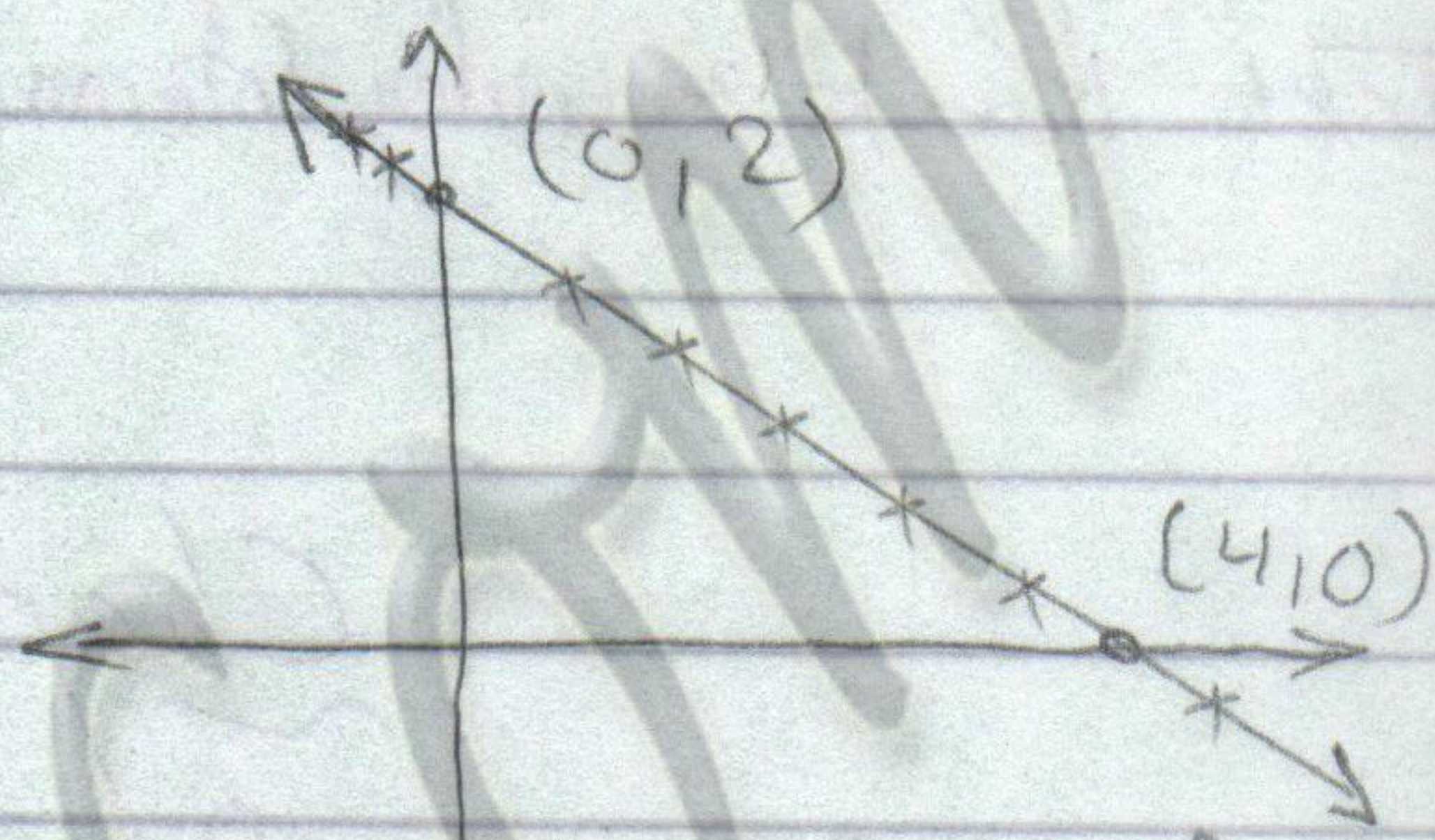
Ex. 1

$$x + 2y = 4$$

Graphically

at $x=0 \rightarrow y=2$ $(0,2)$

at $y=0 \rightarrow x=4$ $(4,0)$



more than one soln.

Using Linear Alge. = let $y=t$ $\therefore x=4-2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Ex. 2 Find the Solution. $x + 2y + 3z = 12$

let $z=t$ $y=5$

$$x = 12 - 2(5) - 3t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

\therefore The Ans. has more than one soln

Ex. 3

$$x + y = 4$$

$$2x + 2y = 6$$

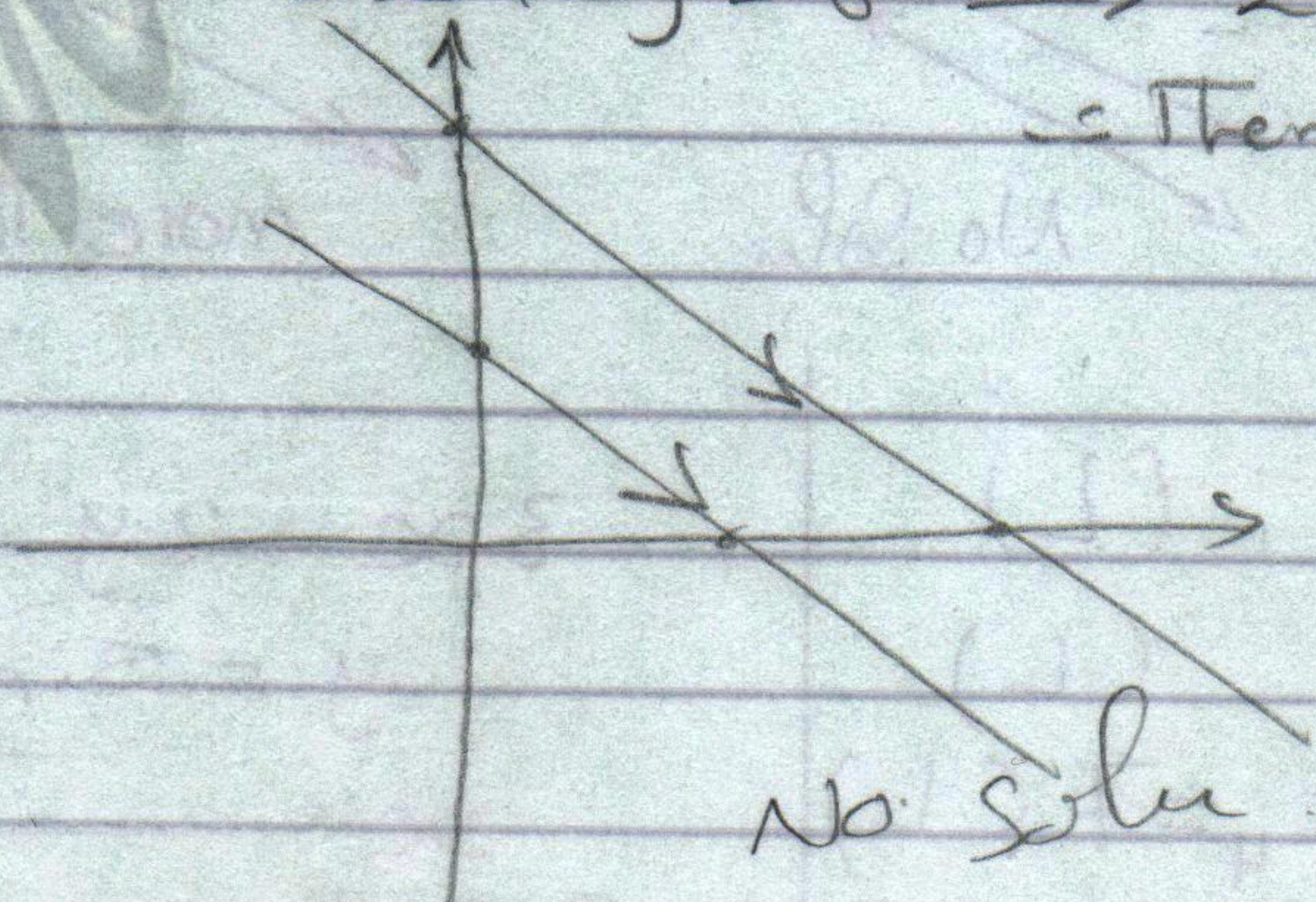
Soln. by Calculus

$$x + y = 4 \rightarrow 1 + y' = 0 \rightarrow y' = -1$$

$$2x + 2y = 6 \rightarrow 2 + 2y' = 0 \rightarrow y' = -1$$

\therefore There is no soln.

by Graph



No soln.

$$\begin{array}{c} -2r_1 + r_2 \rightarrow r_2 \\ \begin{bmatrix} \textcircled{1} & 1 & | & 4 \\ 2 & 2 & | & 6 \end{bmatrix} \end{array}$$

Augmentation Matrix

$$\begin{bmatrix} 1 & 1 & | & 4 \\ 0 & 0 & | & -2 \end{bmatrix}$$

$0 \neq -2$ \therefore There is no Soln.

Ex 4 $x + y = 4$
 $x - y = 4$

by Calculus

$$\rightarrow y' = -1$$

$$\rightarrow 1 - y' = 0 \Rightarrow y' = 1$$

There is only one Soln.

by Graph

1st line $\rightarrow x=0 \rightarrow y=4$, $y=0 \rightarrow x=4$
 2nd line $x=0 \rightarrow y=-4$, $y=0 \rightarrow x=4$

by Li Alg.

$$-r_1 + r_2 \rightarrow r_2 \begin{bmatrix} \textcircled{1} & 1 & | & 4 \\ 1 & -1 & | & 4 \end{bmatrix}$$

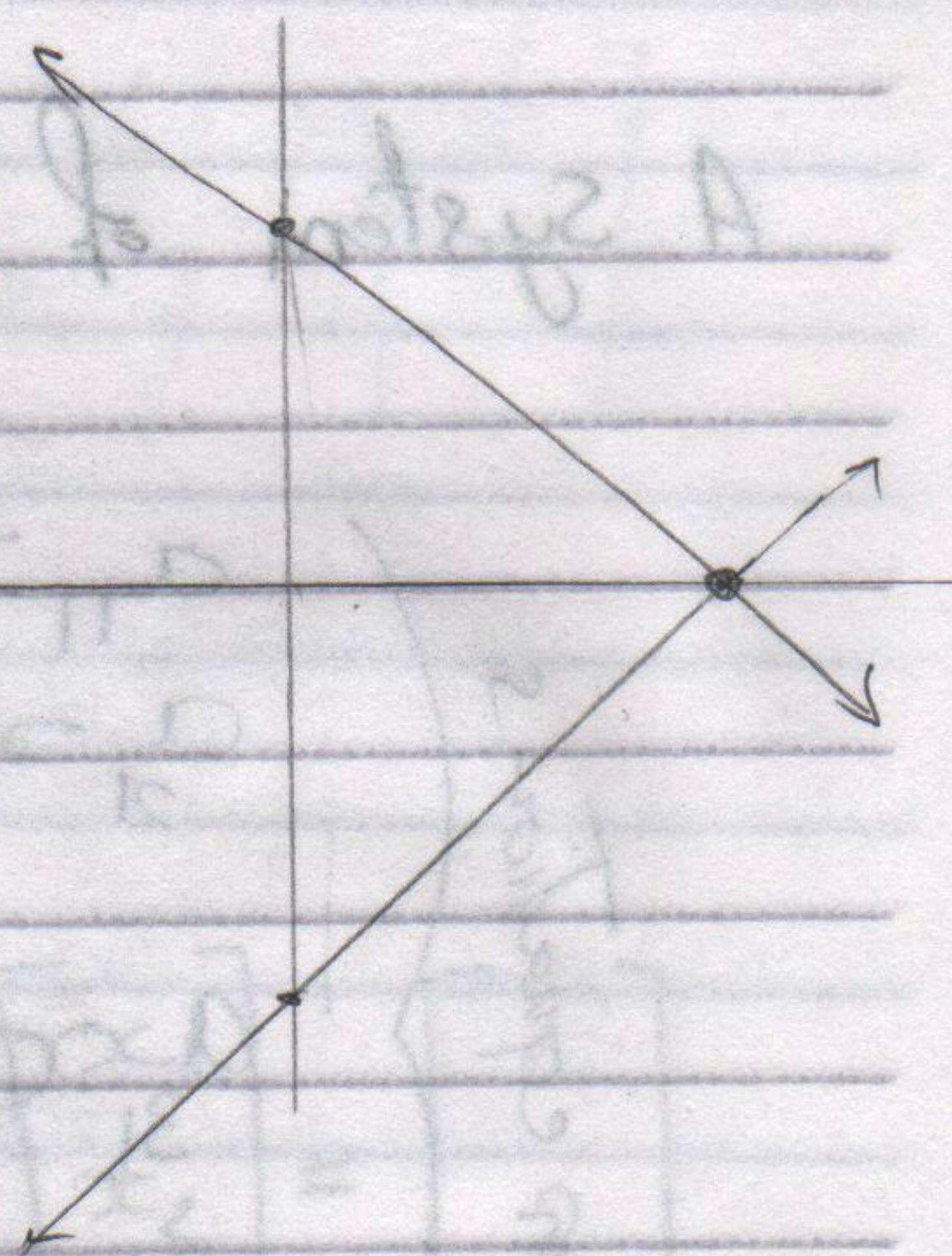
$$\begin{bmatrix} 1 & 1 & | & 4 \\ 0 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$-2y = 0 \Rightarrow y = 0$$

$$x + y = 4 \Rightarrow x = 4$$

\therefore There is only One Soln.



Ex. 5

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 8 \end{aligned}$$

$$-2r_1 + r_2 \rightarrow r_2 \quad \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 2 & 8 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

$$x + y = 4$$

$$\text{let } y = t$$

$$x = 4 - t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

It has more one soln.

Augmented Matrices

A system of m linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{cases}$$

Can be written Only the Rectangular array of numbers =

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & \dots & x_n & \\ a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array} \right]$$

we called that using
Gauss elimination

Ex 6 Find the augmented Matrix for the System of equations

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

$$\begin{array}{l} -2r_1 + r_2 \rightarrow r_2 \\ -3r_1 + r_3 \rightarrow r_3 \end{array} \left[\begin{array}{ccc|c} \textcircled{1} & -3 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 & 1 \\ 3 & 6 & -5 & 0 & 0 \end{array} \right]$$

$$-3r_2 + r_3 \rightarrow r_3 \left[\begin{array}{ccc|c} 1 & 5 & 1 & 2 & 9 \\ 0 & \textcircled{2} & -7 & -17 & -17 \\ 0 & 3 & -11 & -27 & -27 \end{array} \right]$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{array} \right]$$

$$\therefore -x_3 = -3 \Rightarrow \boxed{x_3 = 3}$$

$$2x_2 - 7x_3 = -17$$

$$2x_2 - 21 = -17 \Rightarrow \boxed{x_2 = 2}$$

$$x_1 + x_2 + 2x_3 = 9$$

$$x_1 + 2 + 2(3) = 9 \Rightarrow \boxed{x_1 = 1}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

** We Can Solve the System of Linear Equations by a method called "Elementary row operations", "the Gauss Elimination"

** Three Operations correspond the following operations on the Rows of the augmented Matrix

- ① Multiply a row through by a non Zero Constant
- ② Interchange two rows
- ③ Add a multiple of One row to another row

Ex. 7 Using Element Row Operations (Gauss Elimination) to Solve the System equation.

$$x + y + 2z = 9$$

$$2x + 4y + 3z = 1$$

$$3x + 6y - 5z = 0$$

$$\begin{array}{l} -2r_1 + r_2 \rightarrow r_2 \\ -3r_1 + r_3 \rightarrow r_3 \end{array} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right] \Rightarrow \begin{array}{l} -3r_2 + 2r_3 \rightarrow r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & \textcircled{2} & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{array}$$

$$\therefore -z = -3 \Rightarrow \boxed{z = 3}$$

$$\therefore 2y - 7z = -17$$

$$2y = -17 + 7(3) \Rightarrow \boxed{y = 2}$$

$$\therefore x + y + 2z = 9$$

$$x + 2 + 6 = 9 \Rightarrow x = 9 - 8 \Rightarrow \boxed{x = 1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

\therefore it has just one soln.

(It is same as Ex. 6)

Ex. 8

$$3x_1 - 3x_2 = -2$$

$$2x_1 + x_2 = 1$$

$$3x_1 + 2x_2 = 1$$

$$\begin{array}{l} -2r_1 + 3r_2 \rightarrow r_3 \\ -3r_2 + 2r_3 \rightarrow r_3 \end{array} \left[\begin{array}{ccc|c} 3 & -3 & -2 & -2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & -5 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 3 & -3 & -2 & -2 \\ 0 & 9 & 7 & 7 \\ 0 & 1 & -1 & -1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 3 & -3 & -2 & -2 \\ 0 & 9 & 7 & 7 \\ 0 & 0 & -16 & -16 \end{array} \right]$$

is $0 \neq -16$ \therefore There is no soln.

Ex. 9

$$5x_1 - 2x_2 + 6x_3 = 0$$

$$-2x_1 + x_2 + 3x_3 = 1$$

$$\xrightarrow{2r_1 + 5r_2 \rightarrow r_2} \left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 5 & -2 & 6 & 0 \\ 0 & 1 & 27 & 5 \end{array} \right]$$

let $x_3 = t$

$$\therefore x_2 + 27x_3 = 5 \Rightarrow \boxed{x_2 = 5 - 27t}$$

$$5x_1 - 2x_2 + 6x_3 = 0$$

$$5x_1 - 2(5 - 27t) + 6t = 0$$

$$x_1 = \frac{1}{5} [10 - 54t - 6t]$$

$$\boxed{x_1 = 2 - 12t}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ -27 \\ 1 \end{bmatrix} t + \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

\therefore it has more than one soln.

$$\begin{bmatrix} u \\ v \\ w \\ x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} M + \begin{bmatrix} -16 \\ 0 \\ 11 \\ 4 \\ 0 \end{bmatrix}$$

∴ It has more than one soln.